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High School Teachers' Problem Solving Activities to Review and Extend Their Mathematical and Didactical Knowledge

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Abstract: The study documents the extent to which high school teachers reflect on their need to revise and extend their mathematical and practicing knowledge. In this context, teachers worked on a set of tasks as a part of an inquiring community that promoted the use of different computational tools in problem solving approaches. Results indicated that the teachers recognized that the use of the Cabri-Geometry software to construct dynamic representations of the problems became useful, not only to make sense of the problems statement, but also to identify and explore a set of mathematical relations. In addition, the use of other tools like hand-held calculators and spreadsheets offered them the opportunity to examine, contrast, and extend visual and graphic results to algebraic approaches.

Keywords: Problem solving, high school teachers, community of inquiry, computational tools, and instructional routes.

1. INTRODUCTION

How can high school teachers revise and extend their mathematical and didactical knowledge? To what extent can the use of computational tools help them reconstruct fundamental mathematical relations and results? What types of problems or tasks can help them promote their students' development of mathematical thinking? What features of mathematical thinking should their students develop during their own learning experiences? These types of questions should be addressed in order to find and discuss ways in which practitioners can be involved in academic activities that help them revise and extend their own mathematics and didactical knowledge.

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We argue that there may be different ways for practitioners to enhance their own mathematical ideas and instructional approaches. The identification and discussion of possible paths should be framed around themes that involve mathematics epistemology, problem solving, use of computational tools, and students' learning. Thus, it becomes important for teachers to reflect on ways to characterize the development of the discipline, to analyze the role of problem solving in students' comprehension and development of mathematical concepts, and to discuss the use of distinct computational tools in students' learning. In this context, problem solving activities become crucial for teachers to reflect on their instructional practices. A problem solving approach to construct mathematical knowledge involves a process of constant reflection to refine mathematical ideas.

> . . . *problem solving* is defined as the process of interpreting a situation mathematically, which usually involves several iterative cycles of expressing, testing, and revising mathematical interpretations—and of sorting out, integrating, modifying, revising, or refining clusters of mathematical concepts from various topics within and beyond mathematics [6, p. 782].

It is recognized that there are different paths to examine, characterize, and foster students' development and construction of mathematical knowledge. However, it is important to identify common features that those paths share to promote the students' development of mathematical thinking. Researchers in mathematics education have made significant contributions in areas that include the development of theories to explain the students' construction of new mathematical knowledge, the design of curriculum and mathematical practice, the education and professional development of teachers, the students' comprehension and development of particular domains (numbers, algebra, geometry, probability, etc.), the use of digital tools, and frameworks to evaluate the students' mathematical knowledge [7].

A crucial element to identify and trace the developments of mathematics education as a discipline is the role that mathematical tasks or problems have played in promoting and explaining the students' construction of mathematical knowledge. "In general, a problem arises when we have a *goal*—a state of affairs that we want to achieve—and it is not immediately apparent how the goal can be attained" [3, p. 269]. This typical characterization of a problem emphasizes the need of a goal, and the awareness that there is no an immediate way to achieve it. Indeed, during the problem solving process, new goals and novel forms to attain them often emerge.

This study focuses on analyzing ways in which high school teachers, as a part of an inquiry community, approach mathematical tasks to revise and extend their mathematical and didactical knowledge. That is, the practitioners' process exhibited while working and solving the tasks provides useful information about how these tasks could also be solved by the students [3, p. 268].

2. THE NATURE OF THE TASKS

It is widely recognized that performing mathematical tasks and solving problems play an important role in students' development of mathematical thinking. Indeed, they are considered a key ingredient to frame and support instructional activities that help students foster their understanding of mathematical ideas and the development of problem solving experiences. Mathematical tasks are essential ingredients in any curriculum proposal and their use in the classroom determines or shapes the features of mathematical thinking that students develop or learn [15]. Problems or tasks are a means to provide opportunities for the students to reflect and focus on the development of concepts and problem solving strategies. In this study, a series of tasks was used to promote high school teachers' reflection on ways to approach the problems and use them to prepare instructional routes.

We contend that the mathematical tasks should involve or be placed in a variety of contexts because each context seems to demand different cognitive actions from the problem solver. In this study, the tasks were grouped in terms of the nature of data involved in the problem statement. For example, a task situated in *real context* involved data that come from measuring real objects or parameters associated with actual behaviors of elements of the situation (an example can be found in the appendix). A problem embedded in a *hypothetical context* relies on a series of conditions or assumptions related to the behaviors of data or parameters that are not necessarily real; here, the problem solver has the opportunity to use a variety of resources and mathematical strategies to model it and explore the problem. A problem situated in a *mathematical context* involves essentially dealing with mathematical objects (numbers, lines, points, matrices, etc.) in order to identify, construct, and explore mathematical relations.

In this context, in order for teachers to explore and analyze the didactic potential of a problem, it is important to focus on what they initially identify as relevant in the process of solving the tasks. That is, the qualities of a task can be assessed in terms of analyzing the process of interaction between the problem solver (student) and the task. " . . . [B]eing a 'problem' is not a property inherent in a mathematical task. Rather, it is a particular relationship between the individual [or group] who is trying to solve it" [13, p. 74]. Thus, tasks are the vehicle for students to develop mathematical ideas. "Although the selection of appropriate tasks and tools can facilitate the development of understanding, the normative practices of a class determine whether they will be used for that purpose" [1, p. 26].

We argue that even routine or textbook problems can be of interest to instructors if their solution process does not focus only on getting their solutions, but also on looking for extensions or connections associated with those problems. Thus, it is relevant for teachers to rely on a framework to approach routine problems that can lead students to foster their mathematical inquiry processes [12]. That is, the mathematical content embedded in the task is examined and explored, not only to look for diverse and multiple ways to approach the task, but also to search for connections and extensions of the task.

It is also important for teachers to discuss the students' development of mathematical ideas in terms of identifying possible difficulties experienced while dealing with the task. In addition, teachers should address themes related to instructional approaches, ways to evaluate the students' problem solving performances, and the systematic use of computational tools.

> . . . professional development must provide opportunities for professional growth on the part of teachers and motivate them to develop the knowledge, skills, and dispositions they need to teach mathematics well [14, pp. 160–161].

3. CONCEPTUAL FOUNDATIONS

The conceptual support of the study relies on recognizing that teachers' development of mathematical and pedagogical knowledge could be promoted within an intellectual community that fosters an inquiring or inquisitive approach to mathematical tasks and activities. That is, we recognize the relevance of an inquiring or inquisitive approach for teachers to work on mathematical tasks, and to think of and reflect on their instructional activities. In this context, learning mathematics or developing mathematical knowledge is conceptualized as a set of dilemmas or problems that need to be represented, explored, and solved through the use of mathematical content or resources. The inquiry approach becomes important for the problem solver to develop conceptual understanding, and to solve problems.

> We see understanding as, fundamentally, what results from a person's interpreting signs, symbols, interchanges, or conversations — assigning meaning according to a web of connections the person builds over time through interactions with his or her own interpretations of setting and through interactions with other people as they attempt to do the same [16, p. 99].

Inquiry means for teachers and students to formulate questions, to identify and investigate dilemmas, to search for evidences or information, to discuss solutions, and to present or communicate results [10]. It means willingness to wonder, to pose and examine questions, and to develop mathematical understanding within a community that values both collaboration and constant reflection.

> Inquiry is a multifaceted activity that involves making observations; posing questions; examining books and other sources of information to see what is already known; planning investigations; reviewing what is already known in light of experimental evidence; using tools to gather,

> analyze, and interpret data; proposing answers; and communicating the results. Inquiry requires identification of assumptions, use of critical and logical thinking, and consideration of alternative explanations [8, pp. 13–14].

A mode of inquiry involves necessary challenges to the status quo, and a continuous re-conceptualization of what is learned, as well as how knowledge is constructed.

> [In a community of inquiry] participants grow into and contribute to continual reconstitution of the community through critical reflection; inquiry is developed as one of the forms of practice within the community and individual identity develops through reflective inquiry [4, p. 202].

We posit that the work within the community offers diverse opportunities for its members to search for various ways to approach the tasks, and to identify and discuss relevant concepts and problem solving strategies that can orient the teachers' instructional practices. This process can lead the community to construct new relations, or to identify new routes to reach classic mathematical results. We argue that this type of teacher interaction becomes relevant to foster and to develop a problem solving approach that involves:

> Seeing the mathematical content in mathematically unsophisticated questions, seeing underlying similarity of structure in apparently different problems, facility in drawing on different mathematical representations of a problem, communicating mathematics meaningfully to diverse audiences, facility in selecting and using appropriate modes of analysis ("mental," paper and pencil, or technological), and willingness to keep learning new material and techniques [2, p. 896].

The use of computational tools becomes important for the community, not only to better identify and explore mathematical results, but also to discuss pedagogical paths associated with the hypothetical instructional trajectories that can be useful for teachers to guide or orient their instructional practices.

Mathematical tasks or problems are key elements for teachers to implement curricula proposals [9]. Indeed, tasks and problems form a vehicle that fosters the development of students' reasoning and sense-making activities during mathematical instruction. Teachers as members of a community might move from initial peripheral participation to increasingly more substantial problem solving collaboration [10]. Thus, the community should value and promote all members constantly posing and exploring mathematical questions.

> . . . [this type of approach] places considerable demand on teachers who must organize the physical and normative environment to be conducive to student inquiry, scaffold student actions and thinking as they grapple with

> cognitively challenging tasks, deal with student ideas that are off-track, and, finally, bring closure to a lesson that is rooted in many different student-designed methods of approaching a given task and a variety of levels and kinds of understanding of the to-be-learned concept(s) [15, p. 334].

To work on the problems within an inquiry community offers the in-service teachers the opportunity of constructing potential instructional trajectories that can guide and orient their students' development of mathematical knowledge.

> A climate of inquiry and sense-making, a change in the role of the teacher as the sole authority, and a focus on reasoning and problem solving call for significant changes in a teachers' beliefs about mathematics, about children learning, and about teaching [16, p. 194].

It is also relevant to recognize that the use of a particular tool shapes the way their students could think of the problems [5]; as a consequence, it is important for teachers and students to utilize more than one tool to enhance their ways of thinking of mathematics and problem solving.

3.1. An Inquiry Community

The study is part of a wider project[1] the aim of which is to work with practicing high school teachers on problem solving activities that foster the use of computational tools. "In a community of inquiry, the novice practitioner is drawn into the community through processes of observation, action, questioning of actions, and inquiry into actions" [4]. The community (the participants of the study), was formed by two mathematics educators, one mathematician, one doctoral student, and six high school teachers. The community met once a week, in a three-hour session during one semester, to discuss mathematical tasks and to identify potential routes of instruction that teachers could use to guide and orient the development of their classes. Thus, a problem chosen or designed by one of the members of the community was presented to the group,

[1] This is an ongoing project (supported by two Mexican public institutions: Cinvestav and the Autonomous University of the Hidalgo State titled, Program for Reviewing and Enhancing High School Teachers' Mathematical and Didactic Knowledge through the Use of Computational Technology) that aims to work on and discuss mathematical problems within a community that involves mathematicians, mathematics educators, and teachers. As a community, we are interested in using several computational tools to identify and explore multiple ways to solve the problems. The community's work is used to sketch and discuss possible instructional routes to orient teachers' practices. In this process, all the members of the community have the opportunity to revisit some mathematical results or concepts, and to reflect on ways to organize them for instructional purposes (didactic knowledge).

and all the members worked on the problem to first identify and discuss possible ways to approach the problem. Thus, the participants engaged in an open discussion to make sense of the problem statement, and to explore solution routes. With this strategy, an inquisitive approach to the problem became the dominant method to solve the problem.

In this study, we focus on documenting the approaches that emerged while dealing with a task situated in a real context. That is, the problem statement involves a situation embedded in a real phenomenon. For the initial task, the participants identified pertinent information to simplify and represent the problem.

The unit of analysis of the study was taken as the work shown by the group of participants as a whole, that is, we do not intend to report in detail the participation and contribution or performance of each member, but the approaches shown by the entire community. Thus, the initial work shown by the community was to identify a set of conditions that would translate the statement of the problem into a set of conditions that helped construct a mathematical model. As a result, a dynamic model was constructed. This model was used to analyze the behavior of mathematical relations embedded in the model from graphic, numeric, and algebraic perspectives.

Data that helped us characterize the community's approaches to the problems came from the group's discussion of paired work and individual presentations. All the presentations were videotaped and analyzed initially by the mathematics educators and the mathematician. Later the analysis was shared with the teachers.

During the initial analysis stage, we identified relevant moments where the participants followed up a discussion to explore a particular relation or to examine the consistency or plausibility of an idea suggested by one of the participants. The computer files shown during the presentations, and the notes that the participants gathered during the development of the sessions, were used to characterize the problem solving approaches that emerged in those sessions. That is, the whole group identified and recognized that the process of interaction within the group became crucial to generating a set of relations that were explored during the development of the sessions. Thus, all the participants recognized that the individual and the pairs' contributions were enhanced and complemented during the group's discussions.

3.2. The Task

A truck is approaching a certain underpass where there is a sign indicating the maximum clearance (height of the bridge). The bridge is located just at the base of a descending roadway (Figure 1). What data are essential to know in order to figure it out whether the truck could clear the bridge? What is the effect of the inclination of the roadway on the height of the truck when passing under the bridge?

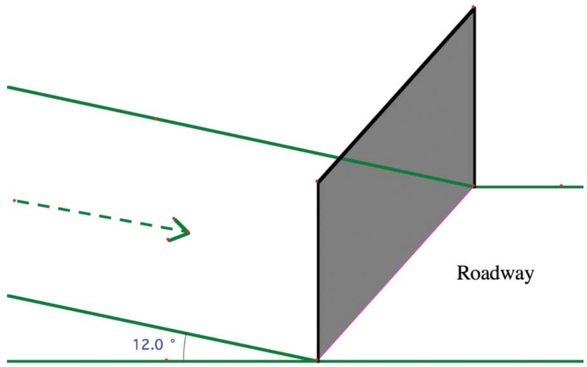

Figure 1. Representation of the roadway and the bridge (color figure available online).

3.3. Initial Considerations

The statement of the task was shown through a LCD projector and the participants also received a written statement. Participants spent some time reading the statement, and then began to ask for more information. What kind of truck should we consider? Where should the wheels be located (at what distance from each other)? What dimensions should the truck's box have? Discussing these types of questions led the participants to think of a two-dimensional representation of the problem in which relevant information was represented in terms of mathematical objects (lines, rays, angles, a rectangle, circles, etc). For example, the roadway was represented through two intersecting lines, the wheels were represented with circles, the truck box with a rectangle, and the height of the bridge with a segment. Figure 2 shows the dynamic model of the problem that was constructed by the group of teachers after analyzing and discussing within the group their individual approaches. Thus, the first goals of the participants were to make sense of the problem statement, and to agree on a set of conditions, that they judged relevant, to represent the problem. For example, they observed that when the positions of the wheels are all on the tilted position or on the horizontal position (after the back wheels have crossed the bridge), the sides AB and DC are parallel to the corresponding lines (roadway). In addition, it was assumed that the truck's tilting effect, which might produce a shifting load on the truck's wheels, does not distort the height of the truck.

3.4. Commentary

The participants recognized that an important stage during the construction of a dynamic representation of the problem was to identify key data, and represent them in terms of mathematical objects. For example, the wheels need to be moved along the tilted and horizontal lines, and this movement needs to

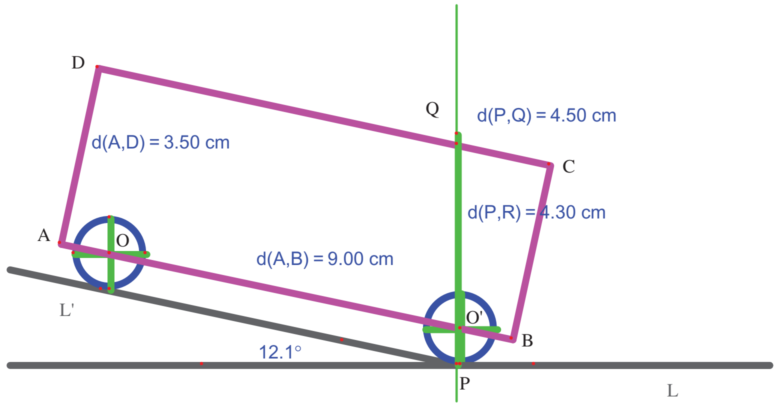

Figure 2. A dynamic model of the problem. The scale used to draw the problem data was 1:100. That is, 1 cm represented by the software represents 100 cm of the original figure. The software allows changing the original dimensions and keeping the scale (color figure available online).

be interpreted in terms of mathematical properties. Thus, the simulation of the motion of the wheel (front wheel, for example), when it touches both the tilted and horizontal part of the roadway, Figure 2) was achieved by drawing a tangent circle to both lines L and L′. Thus, the participants spent time making sense of the problem statement and trying to identify basic relations among the information to represent the statement dynamically.

3.5. Visual Approach

Point M (Figure 3) was a mobile or pivot point used to construct the dynamic representation of the problem. Point Q was chosen at a fixed distance equal to 4.5 cm from P. The participants observed that by moving point M along line L the height of the truck, measured as the distance from point P (bridge initial point) to point R, the intersection point of the perpendicular to line L through point P, and the upper side of the rectangle (segment DC) (Figure 3) change depending on the position of point M. That is, the participants recognized a functional relationship between the position of point M (length of PM) and the corresponding value of PR (height of the truck). Indeed, with the use of the software they showed a graph of the behavior of the truck's height for different positions of point M. Under these conditions, they visualized that there was a position for point M where the height of the truck reaches a maximum value (Figure 3).

In this representation, they observed that when the front wheels are located at M (which is at 3.39 units from the base of the bridge (point P)) then the height of the truck above P reaches 4.61 units. With the help of the software

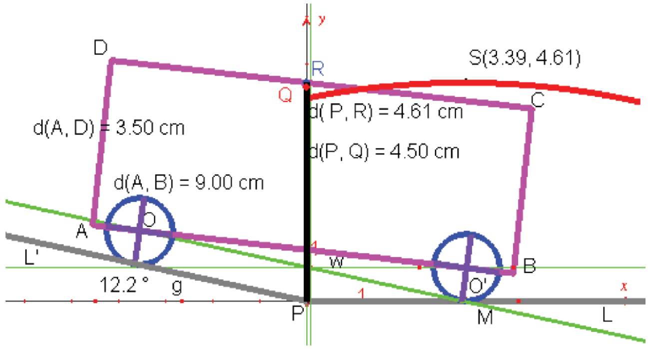

Figure 3. Graphic representation of the variation of segment PR as point M is moved along line L (color figure available online).

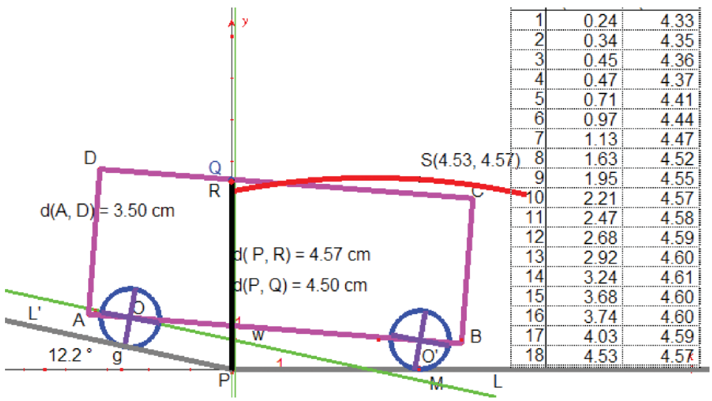

Figure 4. Three representations of the problem: the position of the truck, a table showing the variation of the height, and its graphic representation (color figure available online).

they made a table showing different positions of point M (identified as d(P, M)) and their corresponding height value (d(P, R), Figure 4).

One of the participants proposed to use a spreadsheet program to draw a curve that fits the data. Figure 5 shows the graph of the adjusted curve.

They discussed the importance of constructing a continuous model of the data and used a hand-held calculator to identify the maximum value of the fitted curve (Figure 6).

3.6. Commentary

It was observed that the participants spent a significant amount of time trying to make sense of the problem statement. In particular, they focused on identifying

d(P, M)	d(P, R)
0.24	4.33
0.34	4.35
0.45	4.36
0.47	4.37
0.71	4.41
0.97	4.44
1.13	4.47
1.63	4.52
1.95	4.55
2.21	4.57
2.47	4.58
2.68	4.59
2.92	4.6
3.24	4.61
3.68	4.6
3.74	4.6
4.03	4.59
4.53	4.57

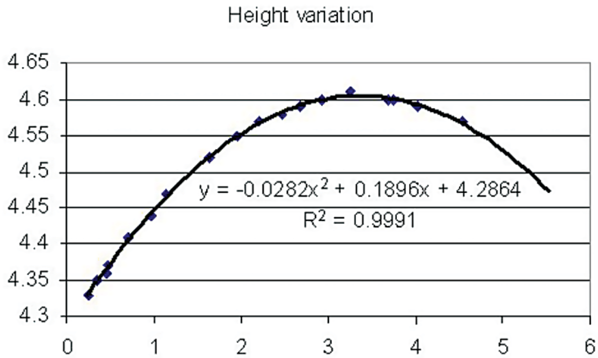

Figure 5. The use of Excel to draw a curve that fits the data (color figure available online).

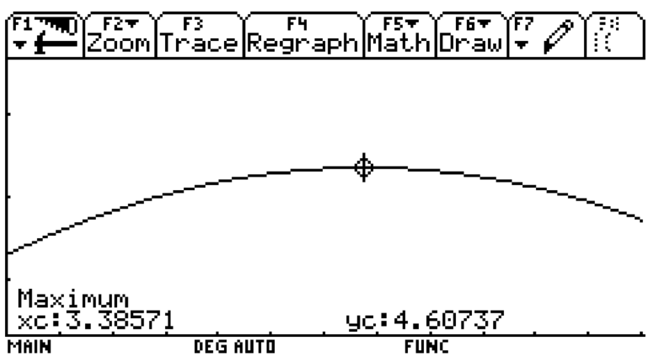

Figure 6. Finding the maximum value of the fitted polynomial.

relevant information to construct a model of the problem. For example, the truck box, wheels, and the highway were represented with a rectangle, circles, and lines, respectively. The dynamic model of the problem was constructed in terms of elements that relate the situation to mathematical objects. Here, the use of the tool does not require expressing symbolically the relation between the position of the truck and its height. It was sufficient to relate the position of the front truck (point M) with the corresponding height (point S), and later the locus of S when point M is moved along line L (the roadway) generating the graph of the variation of the height. By moving the point M along the line L, a table including values for different positions of the point and the associated height of the truck can be produced. The same values were used to construct a graph of an adjusted polynomial with the use of a spreadsheet. This polynomial was also graphed on a hand-held calculator, and the value, where the polynomial reaches its maximum, was shown directly. At this stage, the participants had the opportunity to discuss and contrast results obtained from the dynamic, table, and calculator approaches. They recognized that the use of the software became important to visualize and quantify the variation of the truck height, and to get an approximate value of the maximum height for the truck to go through the bridge underpass. The participants also began to examine the problem representation to construct an algebraic model of the problem.

3.7. An Algebraic Approach

The participants focused on finding an algebraic expression to represent the distance PR (Figure 7) in terms of known data. They observed that the length of segment PR is the sum of lengths of segments PT, TS, and SR.

Based on the information represented in Figure 7, the participants noticed that:

(a) The length of segment PT is the same as the radius of the wheels of the truck.
(b) Triangles $O'ST$ and RSE are similar right triangles, therefore angles ERS and ω ($SO'T$) are congruent.

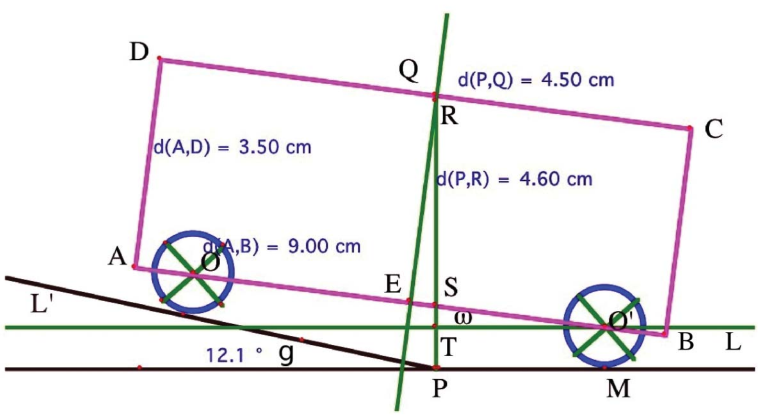

Figure 7. Finding the algebraic model (color figure available online).

(c) $\cos(ERS) = \frac{QE}{RS}$; therefore, $RS = \frac{a}{\cos(\omega)}$, with $a = AD = RE$ and ω the angle $SO'T$.

(d) Also, $\tan(\omega) = \frac{ST}{TO'}$ and since TO' has the same length as $PM = m$, then, $ST = m\tan(\omega)$. This information led them to write:

$$d(P, R) = r + m \tan \omega + \frac{a}{\cos \omega}.$$

They observed that the expression for d(P, R) included two variables (m and ω), and their goal was then to write it in terms of only one of those variables. By examining the figure and introducing others relations, they found the expressions for d(P, R). One in terms of m:

$$d(P, R) = \frac{a}{\cos\left(g - \arcsin\left(\frac{m\sin(g) - r + r\cos(g)}{OO'}\right)\right)} + m\tan\left(g - \arcsin\left(\frac{m\sin(g) - r + r\cos(g)}{OO'}\right)\right) + r.$$

This was the general expression and by substituting the particular values used in the dynamic model ($a = 3.5$; $g = 12.10$; $r = .7$, and $OO' = 7$), they found that:

$$d(P, R) = f(m) = \frac{3.5}{\cos\left(12.1^\circ - \arcsin\left(\frac{m\sin(12.1^\circ) - .7 + .7\cos(12.1^\circ)}{7}\right)\right)} + m\tan\left(12.1^\circ - \arcsin\left(\frac{m\sin(12.1^\circ) - .7 + .7\cos(12.1^\circ)}{7}\right)\right) + .7.$$

They graphed the function $f(m)$ by using a hand-held calculator, and determined the position where the height reaches its maximum value (Figure 8).

The participants noticed that the maximum value of $f(m)$ and that of the quadratic polynomial that fits the data (Figure 5) are almost the same. Thus, they recognized that different approaches to the problem led them to the same result.

They also wrote the expression in terms of ω:

$$d(P, R) = h(\omega) = \frac{a}{\cos \omega} + \left(\frac{OO' \sin(g - \omega) + r - r\cos(g)}{\sin(g)}\right)\tan(\omega) + r$$

$$= \frac{a}{\cos \omega} + \left(\frac{OO'}{\sin(g)}\right)\sin(g - \omega)\tan(\omega) + \left(\frac{r(1 - \cos(g))}{\sin(g)}\right)\tan(\omega) + r.$$

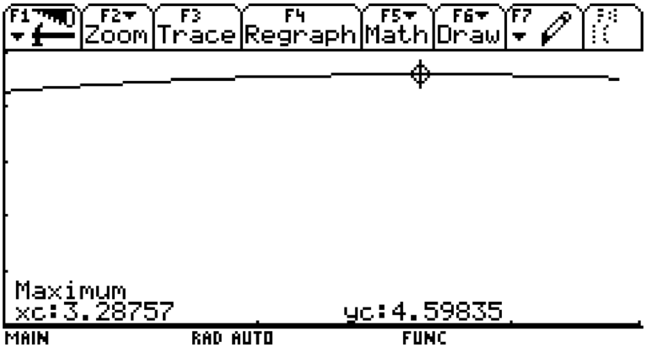

Figure 8. Finding the maximum value graphically.

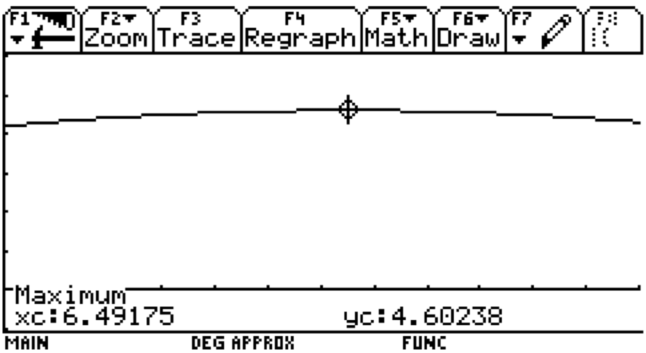

Figure 9. Graphic representation of the expression with particular values.

By substituting the particular values, the graph of the expression was shown in Figure 9.

It was observed that for an angle ω of 6.49 degrees, the length of the segment PR reaches its maximum value of 4.60. With the use of a hand-held calculator, they also determined the maximum value of the height through using calculus techniques. Figure 10 shows that when angle w is 6.49175 the function h(ω) reaches its maximum value.

3.8. Commentary

The participants relied on Figure 7 to pose the question: How can we express algebraically distance PR? To answer this question they tried to write distance PR in terms of the known data of the problem. Basic concepts that they used to represent the distance PR involved the recognition of similar triangles and the

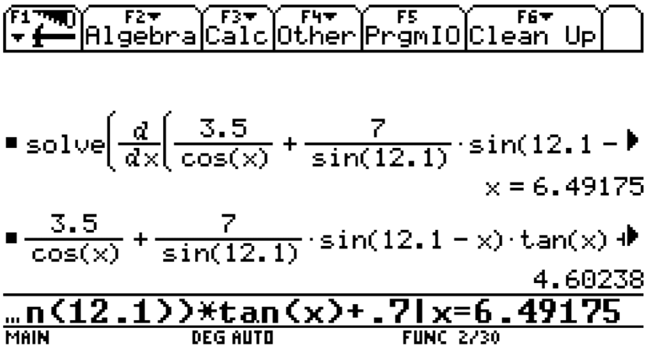

Figure 10. Finding the maximum value with the use of the derivative.

use of trigonometric relations. A crucial stage was to identify the parameters that changed while the truck moved along the roadway. Two parameters (the distance PM $= m$, and the angle w) were identified as variables. It was observed that the participants worked collectively in the search for relations where they could use the available information to express the distance PR in terms of one variable. That is, a suggestion provided by one of the participants was often examined within the group, and the participants made significant contribution to express distance PR in terms of either w or m.

4. DISCUSSION OF RESULTS

The participants recognized that the overarching goals in approaching this task were not only to find and respond to a particular question, but also to identify and explore a set of mathematical relations that emerged during the process of solving the task. It was evident that the use of the tools helped the participants think of different ways to represent and explore mathematical objects associated with the task.

For example, the construction of a dynamic representation of the problem was based on examining mathematical properties associated with the objects used to represent the problem. To this end, they spent a significant amount of time making sense of the problem statement. This phase was relevant to identifying a set of assumptions or conditions to relate elements of the problems to their mathematical representations. For instance, the wheels, and the box of the truck were represented with circles and a rectangle, respectively, while the roadway was represented with a line. The goal in this representation was to model the problem dynamically. The participants chose a point associated with the movement of one of the wheels to simulate the truck motion. To draw the circle tangent to both lines (Figure 11), they used a relation that involved the tangent points, the angle between the two lines, and a trigonometric relation.

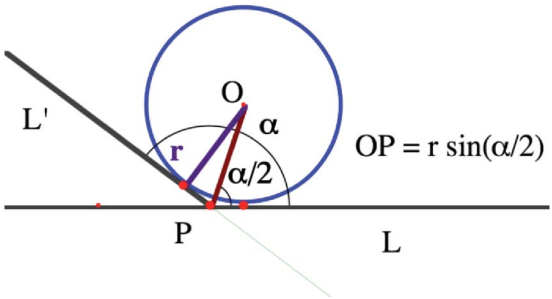

Figure 11. Drawing the tangent to both lines (color figure available online).

Thus, the dynamic representation of the problem became useful not only to visualize the variation of the height of the truck (segment PR), but also to quantify such variation. Here, the participants argued that an advantage in constructing this type of representation is that the variation of the height could be shown graphically or through a table, without expressing the model's variation algebraically. The participants use the spreadsheet tool to analyze the table values that were collected through the software. Their aim was to obtain a polynomial that fit the data, and to identify the maximum value of that polynomial (Figure 5). Here, they also used a hand-held calculator to get the maximum value of the height variation (Figure 6).

The participants recognized that the use of the tools helped them generate the visual and graphic approaches to the problem. These approaches provided them relevant information to comprehend and analyze the mathematical behavior of the height variation, and to identify an approximated value of its maximum; however, they also judged the importance of solving the problem algebraically. To this end, they focused on examining a representation of the problem (Figure 7) in order to express the length of segment PR in terms of known data. Thus, by using mathematical relations associated with similar triangles, right triangles, and trigonometric relations, they initially identified parameters associated with the change of the height. Later, they expressed the behavior of one of those parameter in terms of others elements of the problem. The hand-held calculator was a useful tool for carrying out the needed operation to achieve the algebraic expression. It was important to notice that the participants obtained a general expression for the height (PR segment) that was used to contrast the results obtained previously. That is, they substituted the corresponding values ($a = 3.5$; $g = 12.10$; $r = .7$ and $OO' = 7$) into the general expression and used a hand-held calculator to graph it. On the graph, they identified directly its maximum value. In addition, they used the calculator to get the derivative of the algebraic expression in order to obtain the maximum value.

5. CONCLUSION

Mathematical tasks are key elements for teachers to promote their students' development of mathematical thinking. How should those tasks be examined in order to identify different routes of solutions that help teachers construct didactical paths? We argue that mathematical tasks can be presented and grouped in accordance with the type of data involved in the statement of the problem. We also contend that each group or type of tasks demands that the problem solvers use or apply a series of mathematical resources and strategies in different ways. In addition, teachers should frame their instructional activities in terms of providing their students the opportunity to develop mathematical experiences related to the mathematical, real, and hypothetical contexts.

In this study, a problem that includes actual measurements of roadway steepness (angle), height of a bridge, and a truck's dimensions was used to formulate a question about the truck's clearance of a bridge. The problem also was used to engage high school teachers in an inquisitive approach where they used several computational tools to approach the problem. The direct participation of the mathematics educators, and a mathematician along with the teachers, generated a group atmosphere that helped the participants identify and complement ideas or approaches to the problem. For example, during the initial interaction with the task, for the mathematician the goal was to clearly identify the statement's conditions that helped to simplify and represent the problem. This phase was relevant to thinking of the construction of a dynamic representation of the problem. Here, the teachers and the mathematics educators, who had wide experience in the use of the tools, began to build the geometric configuration to model the problem. From the mathematician's perspective, the relations and results that appeared when the software was used to quantify and explore the variation of certain parameters, needed to be explained and justified in terms of formal arguments. Indeed, this request led the participants to focus on the use of trigonometric relations to construct an algebraic approach to the problem.

The use of a hand-held calculator helped to graph the algebraic model and to identify its maximum value. However, the group also used calculus techniques to examine the algebraic model in order to obtain that maximum value. In general, the participants recognized that the process of solving the task was seen as an opportunity for them to explore various approaches to the problem, and that each approach exhibited diverse mathematical qualities. For example, the empirical approach offered an opportunity to relate the actual physical behavior of the truck to the variation (shown graphically) of its height (Figure 4), while the algebraic model provided the opportunity to relate the problem representation to a set of trigonometric relations. In both cases, issues that involved domains of the relations, dimensions (angles units), and properties of triangles were addressed to construct and analyze the algebraic model. That is, to solve the task not only was it necessary to report the position where the truck reached closest to the bridge, the maximum height; but it was an opportunity for the participants to analyze and connect a set of mathematical concepts. We argue that the open discussion of the tasks within a community that involves teachers, mathematicians, and educators provides a rich environment to first elicit the participants' ideas and later to address the task from diverse angles or perspectives [4, 11]. As a consequence, teachers can revise and extend their mathematical and didactical knowledge to structure and guide their instructional approaches.

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APPENDIX. ANOTHER EXAMPLE OF THE TYPE OF TASKS DISCUSSED DURING THE PROBLEM SOLVING SESSIONS

The task. The figure below shows a car going on a straight roadway. Alongside there is an old church, and the driver wants to stop so that his friend (the passenger) can appreciate the facade of the church. At what point on the road should the driver stop the car, so that his friend can have the best view?

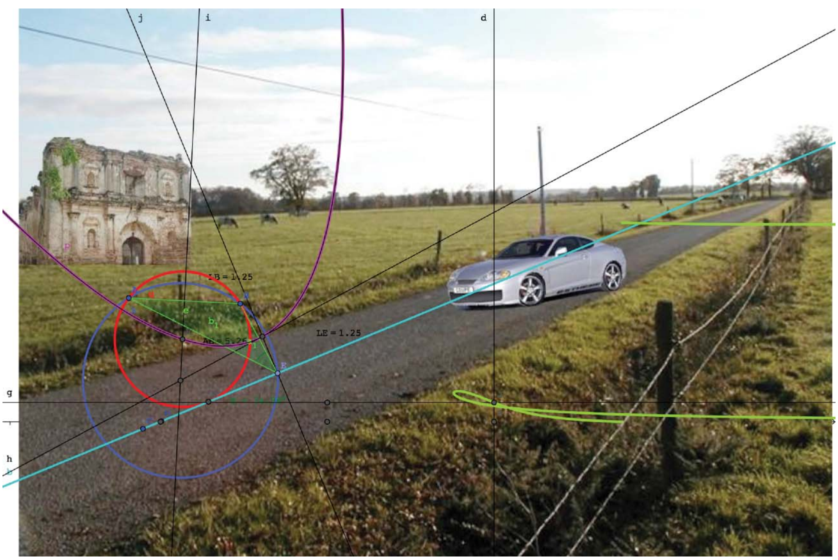

Figure A-1. The church and the car: The geometric figures help determine the point on the road (blue line) where the passanger can get the best view of the church (color figure available online).

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